# BME 50500: Image and Signal Processing in Biomedicine

## Lecture 5: Correlation and Power-Spectrum

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# Content (Lecture Schedule)

**Linear systems in discrete time/space**
Impulse response, shift invariance (4)
Convolution (4)
Discrete Fourier Transform (3)
Power spectrum (7)

**Medial imaging modalities**
MRI (2)
Tomography, CT, PET (5)
Ultrasound (8)

**Engineering tradeoffs**
Sampling, aliasing (1)
Time and frequency resolution (3)
Wavelength and spatial resolution (9)
Aperture and resolution (9)

**Filtering**
Magnitude and phase response (6)
Filtering (6)
Correlation (7)
Template Matching (10)

**Intensity manipulations**
A/D conversion, linearity (1)
Thresholding (10)
Gamma correction (11)
Histogram equalization (11)

**Matlab**

# Cross and Auto-correlation

The cross-correlation is defined as

$$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y^*[n]\,x[n+k]$$

Note that correlation is a convolution with opposite sign. It can be computed with the Fourier transform.

$$R_{xy}(\omega) = Y^*(\omega)\,X(\omega)$$

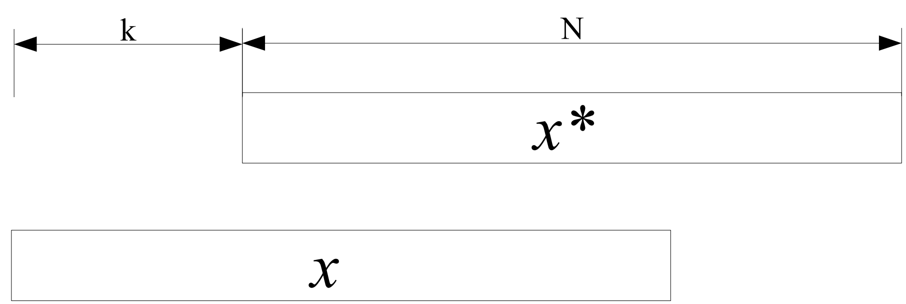
The auto-correlation is defined as

$$r_x(k) = \sum_{n=-\infty}^{\infty} x^*[n]\,x[n+k]$$

# Cross and Auto-correlation

For a sample of finite length $N$ this is typically normalized. We call this the sample auto-correlation

$$\hat{r}_x[k] = \frac{1}{N-k} \sum_{n=1}^{N-k} x^*[n]\, x[n+k]$$




Use `xcorr()` to compute cross of auto-correlation.

# Auto-correlation properties

The **auto-correlation** is symmetric

$$r_x[k] = r_x^*[-k]$$

The zero lag gives the total power of the signal

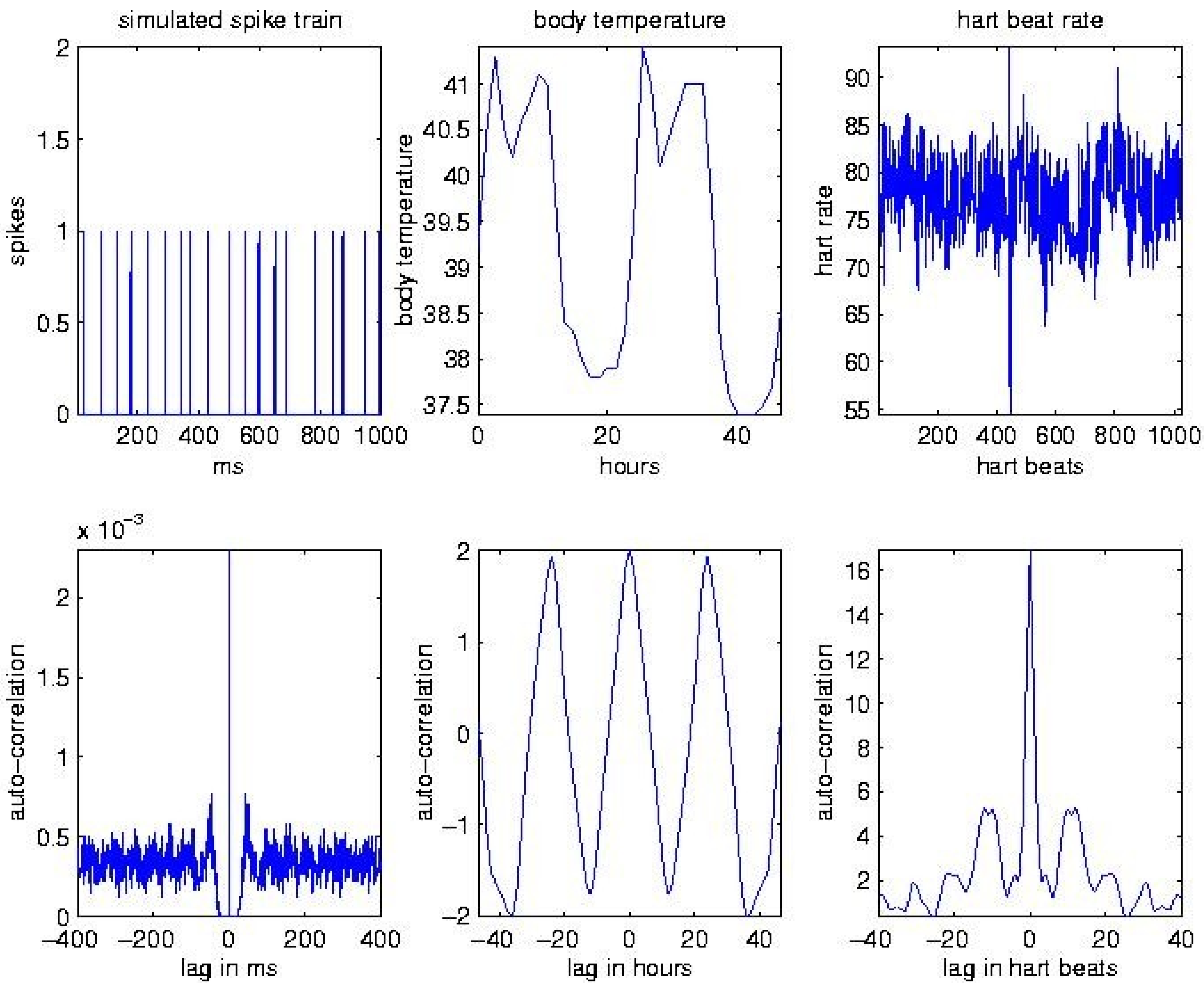
$$r_x[0] = \sum_n |x[n]|^2$$

The auto-correlation has the power as an upper-bound

$$r_x[0] \geq |r_x[k]|$$

# Auto-correlation

Examples



# Power spectrum - properties

Oscillations of the correlation are best analyzed in the frequency domain, which leads to the **Power Spectrum**

$$P_x(\omega) = \sum_{k=-\infty}^{\infty} r_x[k] e^{-jk\omega}$$

One can show that $P_x(\omega)$ is *real*, *even* and *positive*.

The auto-correlation can be recovered with the inverse Fourier transform

$$r_x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega \, P_x(\omega) e^{jk\omega}$$

# Power spectrum - properties

In particular, the total power is given by

$$r_x[0] = \frac{1}{N}\sum_{n=1}^{N} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega\, P_x(\omega)$$

the power spectrum is sometimes called **spectral density** because it is *positive* and the signal power can always be *normalized* to $r(0) = (2\pi)^{-1}$.

Example: Uncorrelated noise has a constant power spectrum

$$r[k] = \sigma^2 \delta(k)$$

$$P_x(\omega) = \sum_{k=-\infty}^{\infty} \sigma^2 \delta(k) e^{-jk\omega} = \sigma^2$$

Hence it is also called **white noise**.

# Effect of Filtering on Power Spectrum

A linear system with impulse response $h[k]$

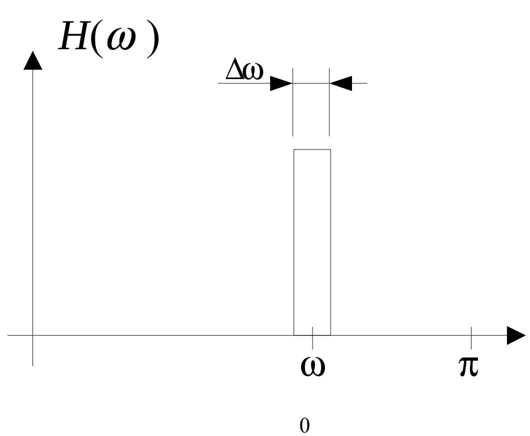
$$x[n] \longrightarrow \boxed{h[k]} \longrightarrow y[n]$$

Transforms the power spectrum as

$$P_x(\omega) \longrightarrow \boxed{|H(\omega)|^2} \longrightarrow |H(\omega)|^2 P_x(\omega)$$

# Spectral Content

"The power spectrum gives the spectral content of the data." To see that consider the power of a signal after filtering with a narrow bandpass filter around $\omega_0$.




$$E\left[|y[n]|^2\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega\, P_y(\omega)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega\, |H(\omega)|^2 P_x(\omega)$$

$$= \frac{1}{2\pi}\int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} d\omega\, P_x(\omega)$$

$$\approx \frac{\Delta\omega}{2\pi} P_x(\omega_0)$$

# Power spectrum - properties

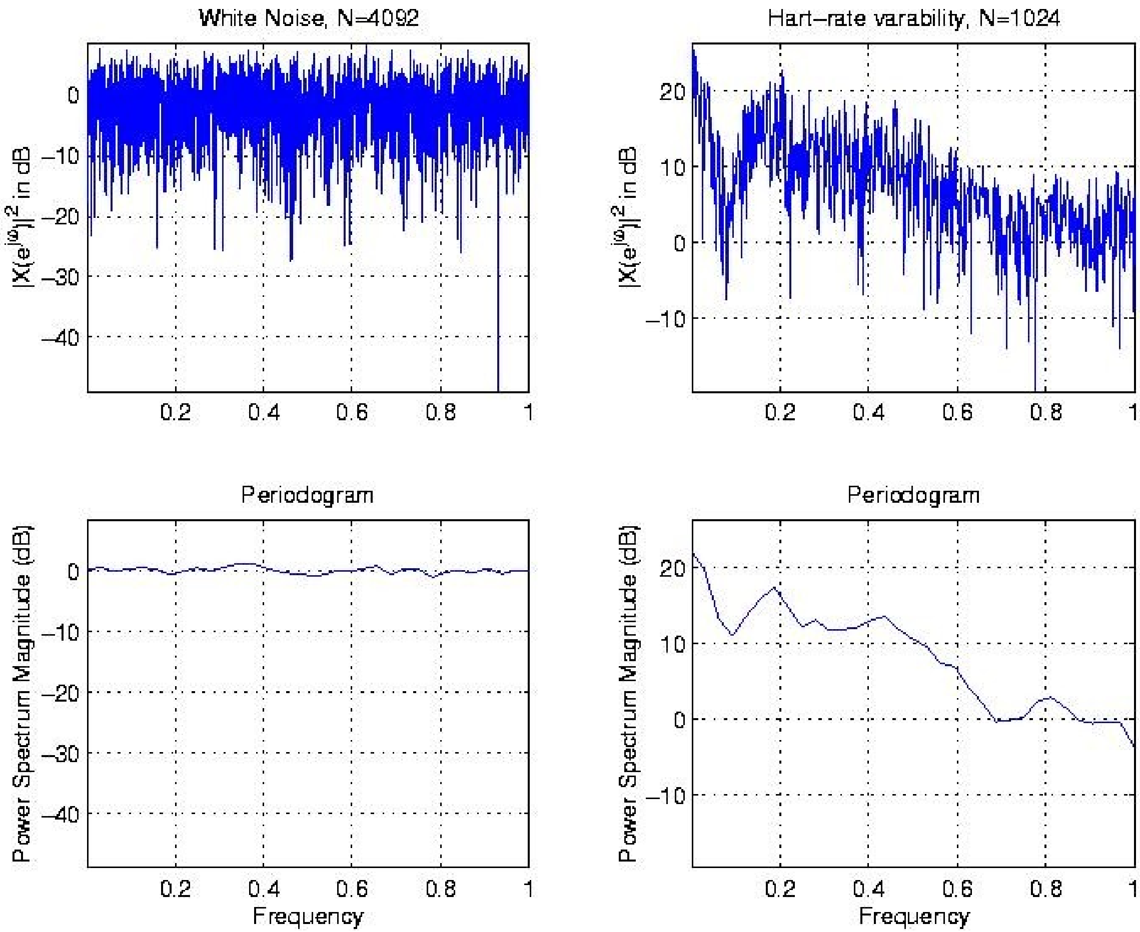
The power spectrum captures the spectral content of the sequence. It can be estimate directly from the Fourier transform of the data:

$$\hat{P}_x(\omega) = \frac{1}{N} |X(\omega)|^2$$

$$X(\omega) = \sum_{k=0}^{N-1} x[k] e^{-j\omega k}$$

# Power spectrum

Unfortunately, the direct estimate is inconsistent, i.e. its variance does not converge to 0 for increasing N.

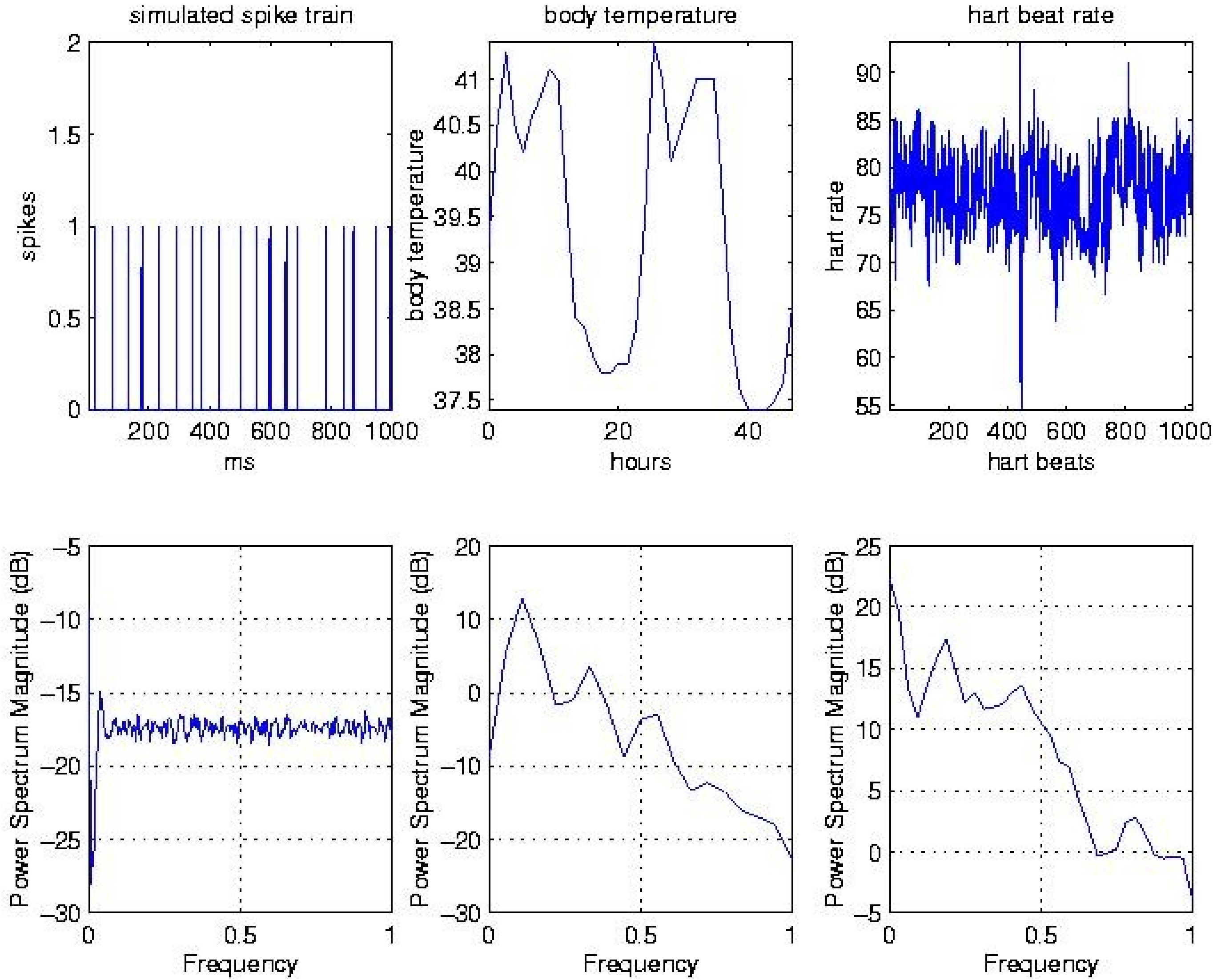



A classic heuristic, called the **periodogram**, is to smooth neighboring frequencies: Compute Fourier transform for window of size N/K and average over K windows.

# Periodogram

Examples:

```
>> psd(x);
```

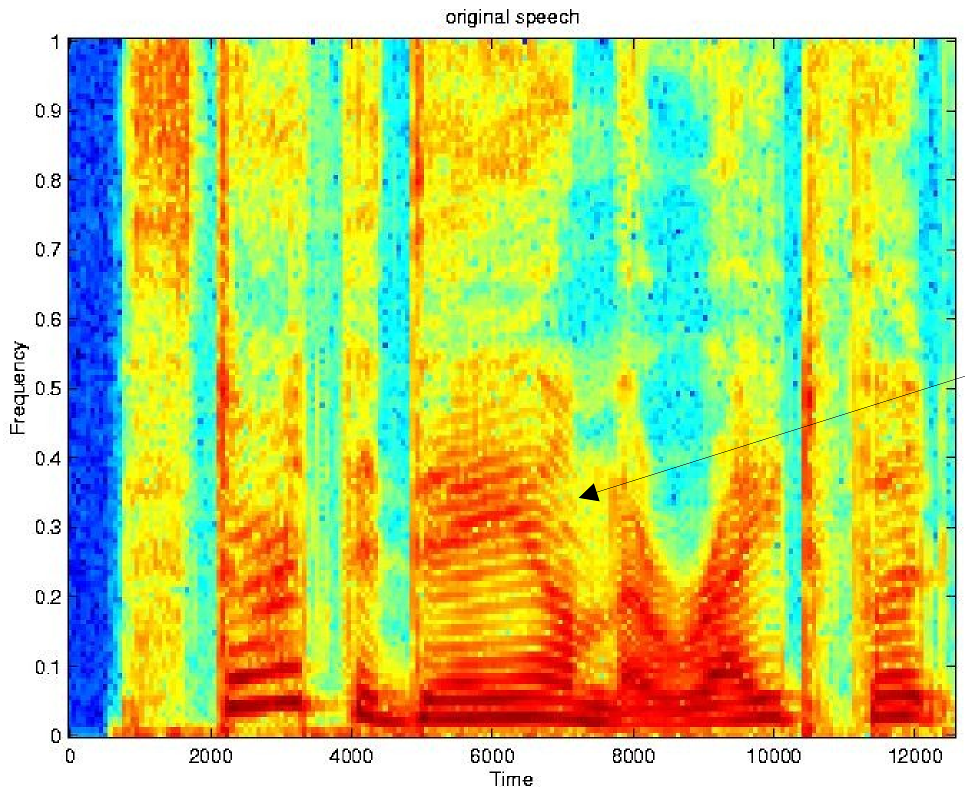



Note 1/f spectrum, noise for small N.

# Spectrogram

**Spectrogram**: Power spectrum with strong spectral component can be estimate on short sequences, and hence, followed as it develops over time.

Example: Speech `>> specgram(x);`

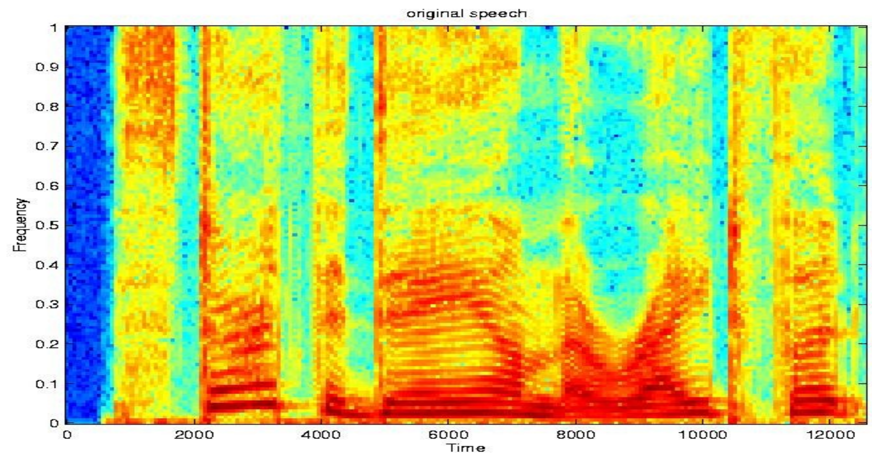



Note harmonic components

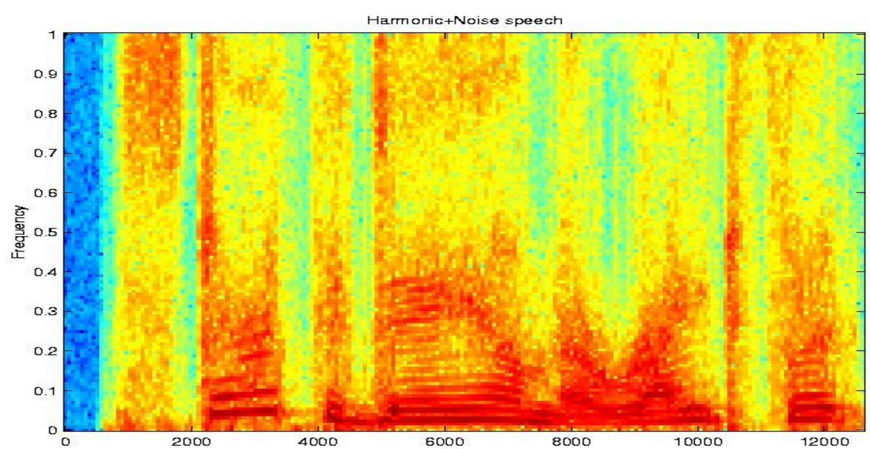
# Harmonic Analysis

Speech can be modeled as harmonic component plus colored noise process.

Original




Synthesized




Harmonic

Noise

# Assignment

**Assignment 8**: **Detect and measure alpha activity in EEG**

- Load a EEG signal and determine the exact frequency of “alpha activity” (approximately 10 Hz oscillation) by computing the power spectrum averaged over the 338 trials and averaged over all 64 electrodes.
- Pick a electrode that has strong power in the “alpha band”. Display the power spectrum for this electrode.
- Design a filter to extract only this alpha band activity. Display magnitude and phase response of the filter. Filter the signal avoiding delays. For the electrode you have picked, show the filtered signal in the time domain. You could display it as an image of samples by trials, or as a regular curve plot. Useful function when dealing with 3D volumes is squeeze().
- Design a filter to extract everything but the alpha activity.  Display magnitude and phase response. Show the filtered signal in the time domain as before for the same electrode.
- Determine the signal-to-noise ratio of the alpha activity for each electrode and display its distribution over the head.