BME 50500: Image and Signal Processing in Biomedicine

Lecture 2: Discrete Fourier Transform

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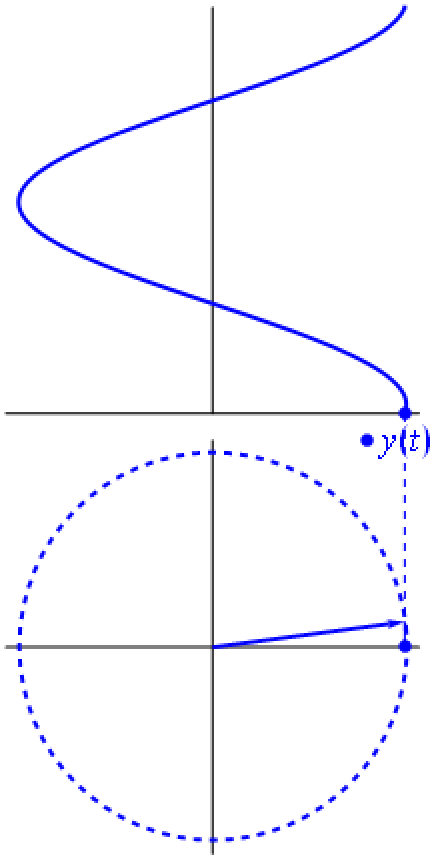
Complex Numbers

Complex numbers simplify mathematical analysis of time-varying quantities (i.e., the signals that we want to analyze).

$$A\cos(\omega t) = \frac{A}{2}\exp(i\omega t) + \frac{A}{2}\exp(-i\omega t)$$

$$A\cos(\omega t) = \Re\{A\exp(i\omega t)\}$$



A complex sinusoid may be expressed in phasor notation

$$z = |z|\exp(i\omega t) = |z|(\cos\omega t + i\sin\omega t)$$

Continuous Fourier Transform

The Fourier transform is a functional that takes a function of time $f(t)$ and maps it to a function of frequency $F(\nu)$

$$F(\nu) = \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i 2\pi\nu t} \equiv F\{f(t)\}$$

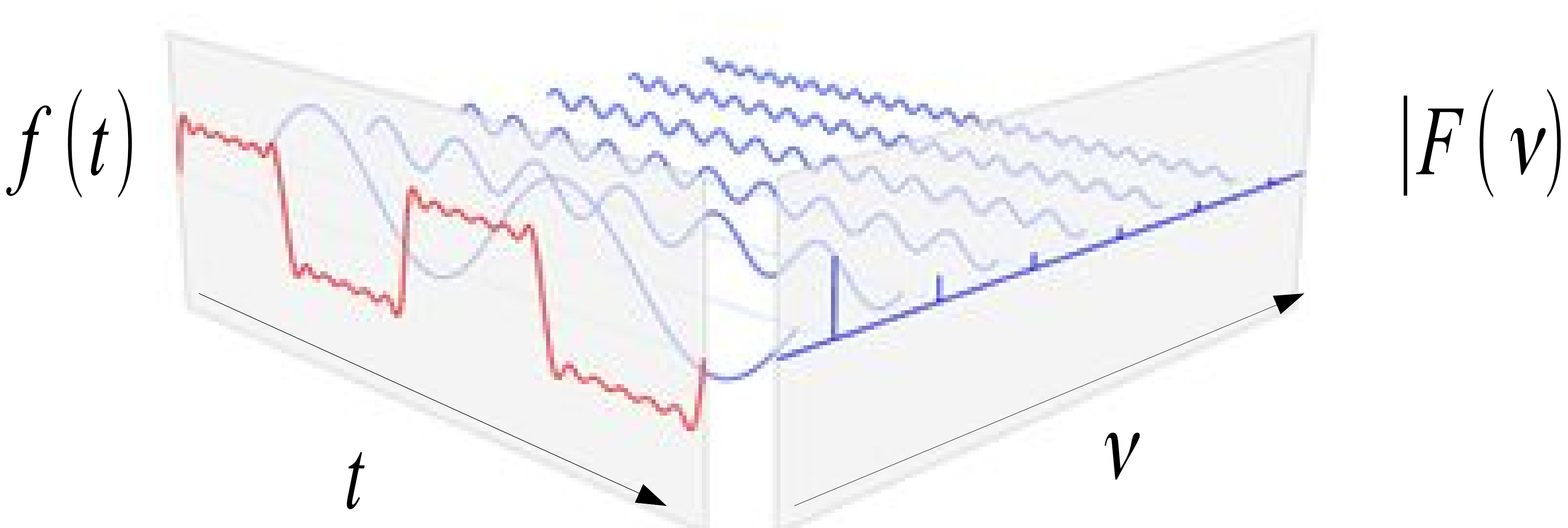
It has an **inverse transform** that can recover the 'time domain' function from it 'frequency domain' transform.

$$f(t) = \int_{-\infty}^{\infty} d\nu\, F(\nu)\, e^{i 2\pi\nu t}$$

In this transform pair both time and frequency are continuous. In signal and image processing however time and frequency may have to be discretized.

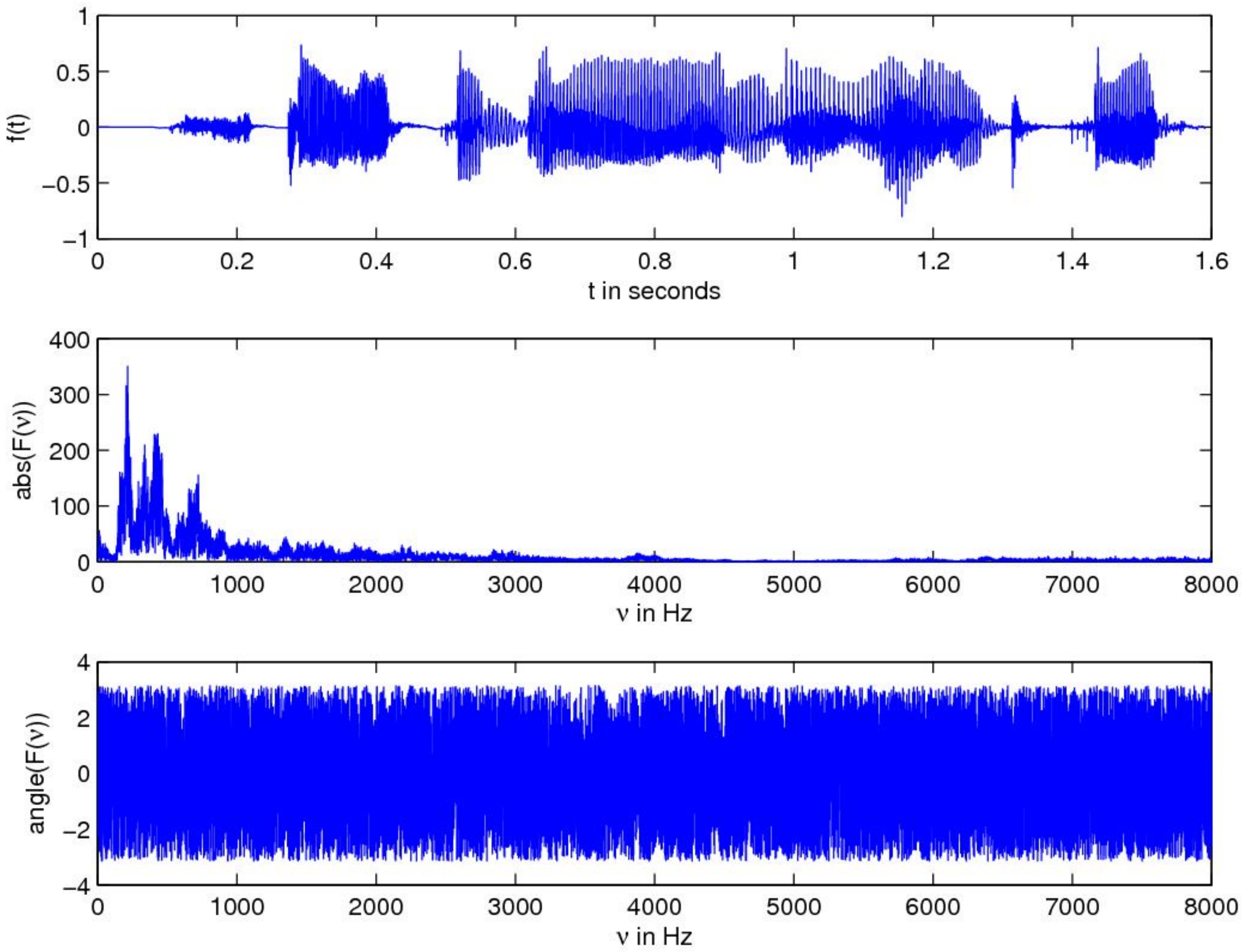
Continuous Fourier Transform

By Lucas V. Barbosa https://commons.wikimedia.org/w/index.php?curid=24830373



Continuous Fourier Transform

Example of a signal and its Fourier decomposition



CFT - Examples

Examples

$$FT\{1\} = \delta(\nu)$$

$$FT\{\delta(t - t_o)\} = e^{-2\pi i \nu t_o}$$

$$FT\{e^{-at^2}\} = \sqrt{\frac{\pi}{a}}\, e^{-\pi^2 \nu^2 / a}$$

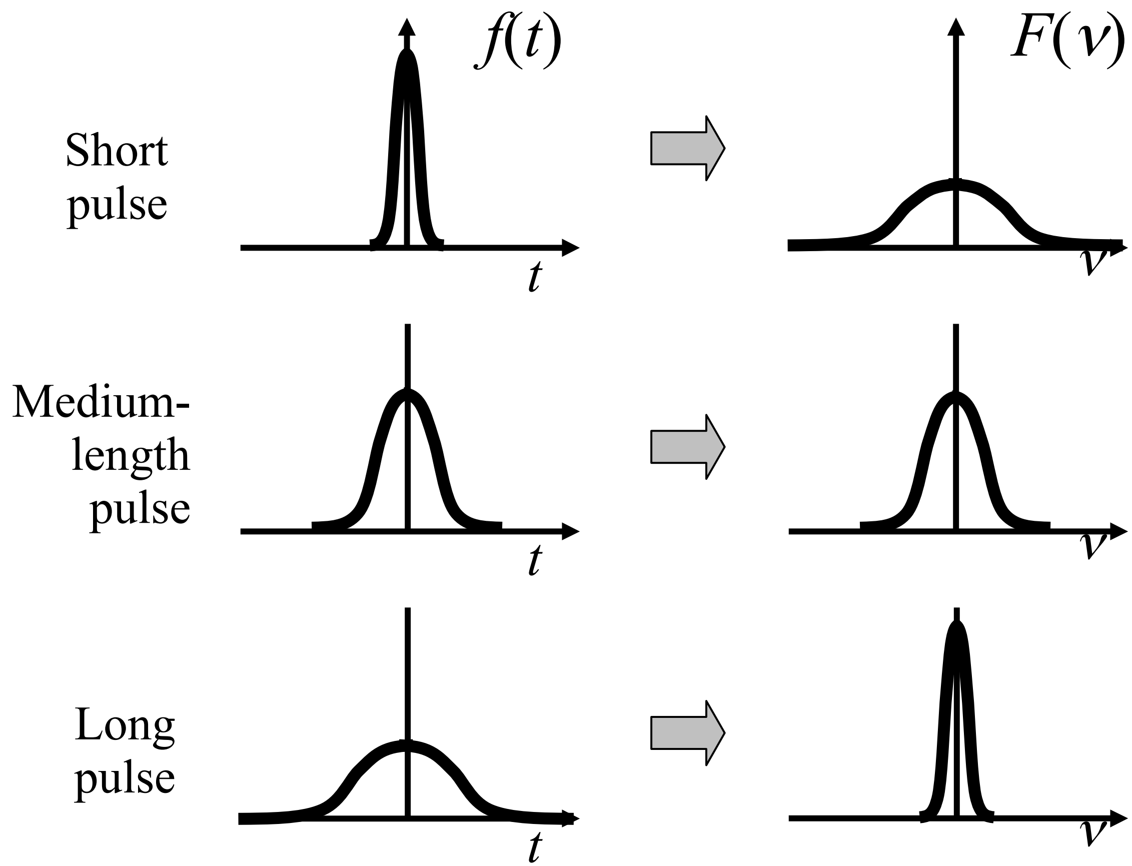
$$FT\{\cos(2\pi \nu_o t)\} = \frac{1}{2}(\delta(\nu - \nu_o) + \delta(\nu + \nu_o))$$

$$FT\{rect(t)\} = sinc(\nu) = \frac{\sin(\pi \nu)}{\pi \nu} \qquad rect(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & else \end{cases}$$

CFT – Scale Theorem

'Stretching' time 'shrinks' frequency and vise versa

$$FT\{f(a\,t)\} = \frac{1}{a} FT\left(\frac{\nu}{a}\right)$$



The shorter the pulse, the broader the spectrum!

This is the essence of the Uncertainty Principle!

CFT – Scale Theorem derivation

(1) From the definition:

$$f(t)=\int_{-\infty}^{\infty} F(\nu)e^{i2\pi\nu t}d\nu$$

(2) It then follows that:

$$f(at)=\int_{-\infty}^{\infty} F(\nu)e^{i2\pi\nu at}d\nu$$

(3) Change of variables

$$\omega=\nu a \Rightarrow d\omega=a\,d\nu$$

(4) Substituting into the integral:

$$f(at)=\int_{-\infty}^{\infty} \frac{1}{a}F\left(\frac{\omega}{a}\right)e^{i2\pi\omega t}d\omega$$

$$FT\{f(at)\}=\frac{1}{a}FT\left\{\frac{\nu}{a}\right\}$$

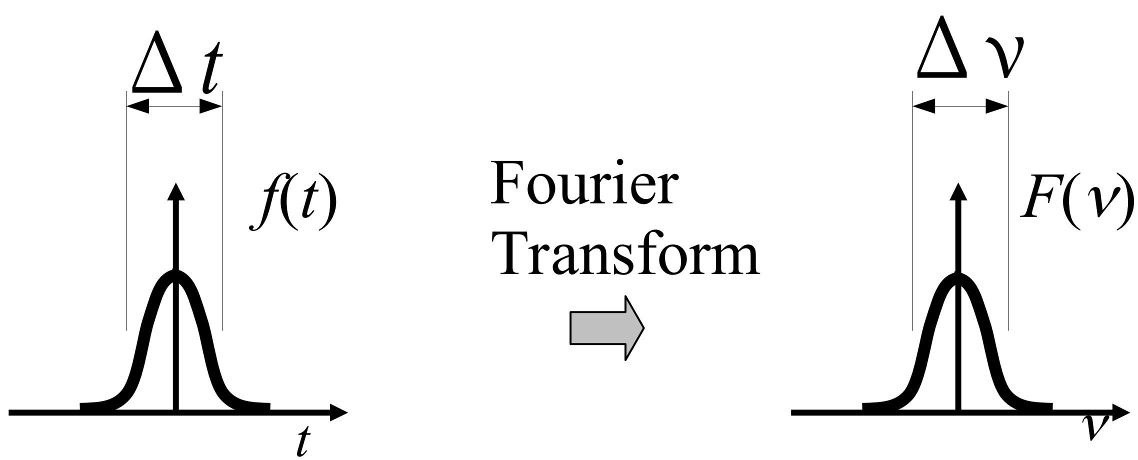
CFT – Uncertainty Principle

There is a tradeoff between ***temporal extend*** and ***frequency bandwidth***. There is a lower bound on the time-bandwidth product

$$\Delta t \, \Delta \nu \geq \frac{1}{2}$$

$$\Delta t^2 = \frac{1}{B} \int dt \left| f(t) \right|^2 t^2 \qquad \Delta \nu^2 = \frac{1}{B} \int d\nu \left| F(\nu) \right|^2 \nu^2$$

$$B = \int d\nu \left| F(\nu) \right|^2 = \int dt \left| f(t) \right|^2$$



- A signal that has a well defined frequency must be extended in time.
- A short signal must be broadband.

Discrete Time Fourier Transform (DTFT)

The **Discrete Time Frourier Transform** (DTFT) is defined on the unit circle $z = e^{j\omega} = \cos(\omega) + j\sin(\omega)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega}$$

Angular frequency

$$\omega = 2\pi\nu$$

The DTFT is an invertible transformation

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega\, X(e^{j\omega}) e^{jn\omega}$$

This is simply because $\int_{2\pi} d\omega\, e^{-j\omega n} = 2\pi\,\delta_n$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega\, X(e^{j\omega}) e^{jn\omega} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} d\omega\, e^{-j\omega(k-n)} = x[n]$$

DTFT - Properties

The following properties can be derived from its definition:

Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Delay	$x[n-n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
Time reversal	$x[-n]$	$X(e^{-j\omega})$

Correlation

$$\sum_{n=-\infty}^{\infty} x[n+k]\, y^*[n] \qquad X(e^{j\omega})\, Y^*(e^{j\omega})$$

Conjugate symmetry for $x[n] \in \mathbb{R}$ $\qquad X(e^{j\omega}) = X^*(e^{-j\omega})$

Discrete Fourier Transform (DFT)

The **Discrete Fourier Transform** (DFT) is the **sampled** DTFT

$$X(e^{j\omega})_{\omega=2\pi k/N} = X[k]$$

The ***N*-point DFT** is defined for a signal of length N : (analysis)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j 2\pi kn/N}$$

with **inverse *N*-point DFT**: (synthesis)

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j 2\pi kn/N}$$

Note that specifying $X[0] \ldots X[N\text{-}1]$ implies a *synthesized* periodic signal outside $n = 0 \ldots N\text{-}1$:

$$x[n] = x[n \bmod N]$$

Discrete Fourier Transform - FFT

In signal processing we always work with the DFT since we can compute Frourier transform only for discrete frequencies.

Important result on computational cost: While computing DFT values $X[k]$, $k=1 \ldots N$, would seem to take N^2 operations there is an efficient method called **Fast Fourier Transform** (FFT) of order:

$$N \log_2 N$$

```
Matlab:
>> X =  fft(x);
>> x = ifft(X);

Python:
import numpy.fft as fft
X = fft.fft(x)
```

With this one can implement convolution in $\log_2(P)$ operations per sample rather than P.

Fourier Transform Summary

Fourier transform
(con. time, cont. freq.)

$$X(\nu)=\int_{-\infty}^{\infty} dt\, x(t)\, e^{-j2\pi\nu t}$$

$$x(t)=\int_{-\infty}^{\infty} d\nu\, X(\nu)\, e^{j2\pi\nu t}$$

Discrete time FT
(disc. time, con. freq.)

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega}$$

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega\, X(e^{j\omega})\, e^{jn\omega}$$

Fourier series
(con. time, disc. freq.)

$$a_k=\frac{1}{2T}\int_{-T}^{T} dt\, x(t)\, e^{-jk\omega t}$$

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

Discrete FT
(disc. time, disc. freq.)

$$X[k]=\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N}$$

Fourier Transform

Assignment 2:

- Generate a real-valued (sampled) sinusoid signal and display as a function of time (in seconds).
- Compute the DFT of the signal and display in the same figure the real and imaginary parts of the DFT as well as its magnitude and phase all as a function of frequency. Also show the original signal in the time domain – a total of 5 graphs in one figure. Label the frequency axis in Hz.
- Test empirically which is the highest frequency you can represent at a fixed sampling rate. Show an example above and bellow that frequency in the time domain and magnitude in the frequency domain – a total of 4 graphs.
- Generate a complex valued sinusoid with negative frequency; display it in the time domain (real and imaginary parts) and frequency domain (real and imaginary parts) – a total of 4 graphs. Contrast that to a sinusoid of positive frequency – another 4 graphs.
- Generate a sinusoidal signal at some frequency f plus a sinusoid at 2f (added together) and downsample it to a new sampling rate of fs=3.5*f while avoiding aliasing. Display the absolute value of the DFT before and after down-sampling. Label the frequency axis in Hz.

Sampling - Sampling Theorem*

What is the relation between a continuous time signal $x(t)$ and its sampled version $x[n]$?
To answer that, compare the continuous time Fourier transform (CTFT) of a continuous time signal $x(t)$ given by,

$$X_c(\Omega)=\int_{-\infty}^{\infty} dt\, x(t)\, e^{-j\Omega t}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\Omega\, X_c(\Omega)\, e^{j\Omega t}$$

and the DTFT, $X(e^{j\omega})$ of the signal sampled at frequency $f_s=1/T$, $x[n] = x(nT)$?

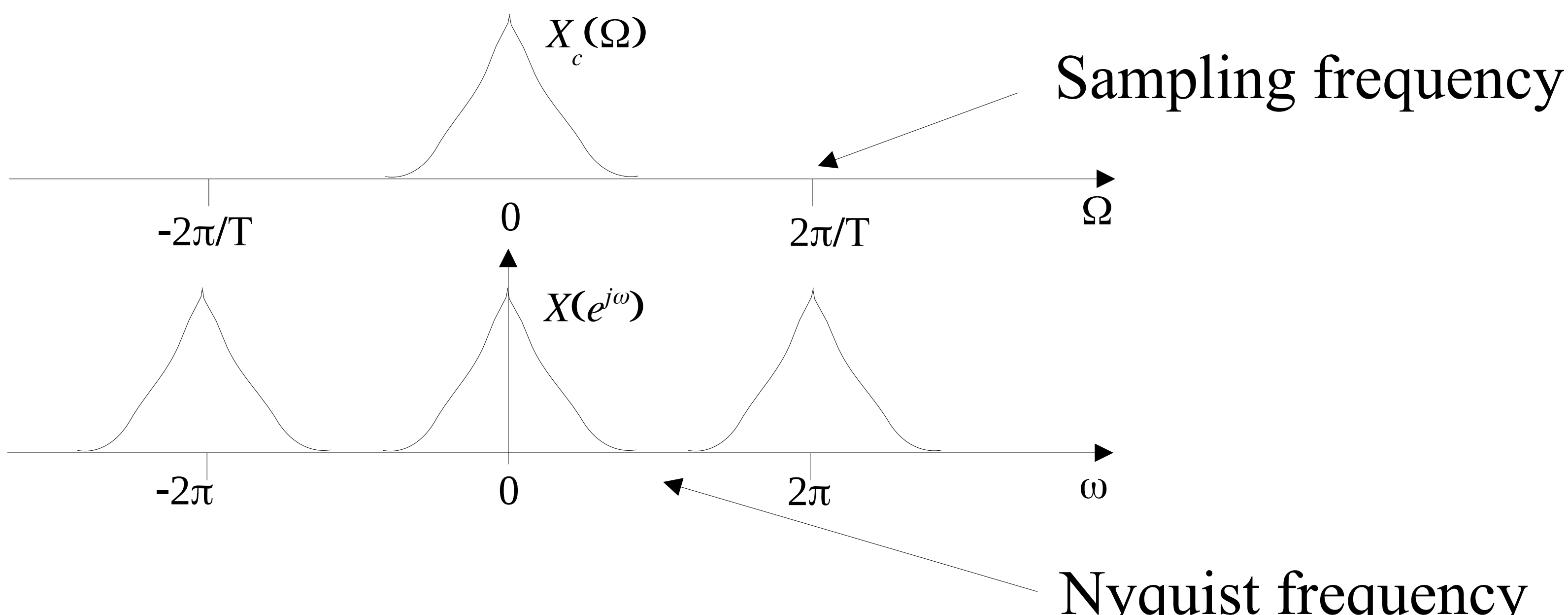
$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega}$$

Sampling - Sampling Theorem

According to the Sampling Theorem the relation is:

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)$$

The DTFT repeats the CTFT with a period 2π.



Contributions above π will overlap with different period!

Sampling - Sampling Theorem

Under which conditions can we determine continuous time $x(t)$ from discrete time $x[n]$, $t=nT$?

If the signal is *bandlimited:* $X_c(\omega/T)=0, \quad |\omega|\geq\pi$

Then we can determine $X_c(\Omega)$ from $X(e^{j\omega})$ according to the Sampling Theorem:

$$X(e^{j\omega})=\frac{1}{T}X_c\left(\frac{\omega}{T}\right)$$

In that case we can determine x[t] $\rightarrow$ $X(e^{j\omega})$ $\rightarrow$ $X_c(\Omega)$ $\rightarrow$ x(t).

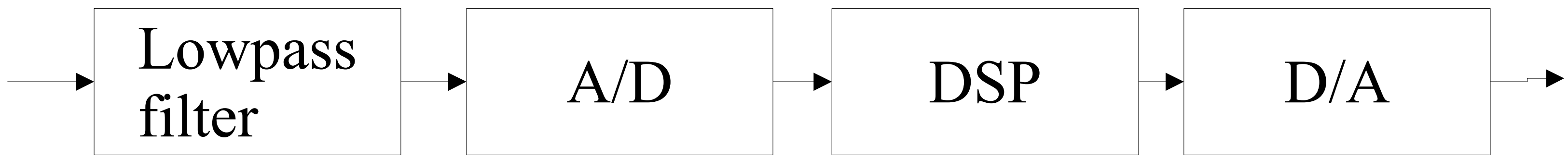
After some algebra:

$$x(t)=\sum_{n=-\infty}^{\infty}x[n]sinc\left(\frac{t-nT}{T}\right)$$

$x(t)$ is $x[n]$ convolved with $sinc(t)=\sin(\pi t)/(\pi t)$

Sampling

Because of the sampling theorem always!:



Make sure you lowpass filter the signal to half the sampling frequency (Nyquist) before you sample.

If you can not filter prior to sampling make sure that you choose the sampling frequency to be twice the highest frequency that contains significant signal power.

Do not down sample by simply taking every other sample. First lowpass filter then subsample. Better yet, use either

```
>> x = resample(x,P,Q);
>> x = decimate(x,Q/P); % or scipi.decimate()
>> x = downsample(x,Q/P) % don't use this!!
```

Sampling – what can go wrong

If one fails to low-pass filter before sampling, then frequencies above Nyquist “leak” into the sampled frequency band.

Example: Low-frequency sinusoid (10 Hz) plus noise that is only high-frequency (only components above 25Hz). When sampled at 50Hz without previously removing the high frequency components the HF noise appears at lower frequencies below 25Hz, i.e. it is “reflected” or “leaks” from the high to the low frequencies.